PRODUCTS OF CHARACTERS AND FINITE p-GROUPS III

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ABSTRACT. Let G be a finite p-group and χ, ψ be irreducible characters of G. We study the character $\chi\psi$ when $\chi\psi$ has at most $p-1$ distinct irreducible constituents.

1. Introduction

Let G be a finite p-group. Denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. Let $\chi, \psi \in \mathrm{Irr}(G)$. Then the product of $\chi\psi$ can be written as

$$\chi\psi = \sum_{i=1}^{n} a_i \theta_i$$

where $\theta_i \in \mathrm{Irr}(G)$ and $a_i = [\chi\psi, \theta] > 0$. Set $\eta(\chi\psi) = n$. So $\eta(\chi\psi)$ is the number of distinct irreducible constituents of the product $\chi\psi$. The purpose of this note is to study the case when the product $\chi\psi$ has "few" irreducible constituents, namely when $\eta(\chi\psi) < p$.

If χ is a character of G, denote by $\mathrm{V}(\chi) =< g \in G \mid \chi(g) \neq 0 >$. So $\mathrm{V}(\chi)$ is the smallest subgroup of G such that χ vanishes on $G \setminus \mathrm{V}(\chi)$. Through this work, we use the notation of [3]. In addition, we are going to denote by $\mathrm{Lin}(G) = \{\lambda \in \mathrm{Irr}(G) \mid \lambda(1) = 1\}$ the set of linear characters. The main results are

Theorem A. *Let G be a finite p-group and $\chi, \psi \in \mathrm{Irr}(G)$. Assume that $\eta(\chi\psi) < p$. Let $\theta \in \mathrm{Irr}(G)$ be a constituent of $\chi\psi$. Then*

(i) $\mathbf{Z}(\chi\psi) = \mathbf{Z}(\theta)$.

(ii) $\mathrm{V}(\chi\theta) \geq \mathrm{V}(\theta)$. Therefore $\mathrm{V}(\chi) \cap \mathrm{V}(\psi) \geq \mathrm{V}(\theta)$.

Theorem B. *Let G be a finite p-group and $\chi, \psi \in \mathrm{Irr}(G)$. Assume that $\eta(\chi\psi) < p$. Let N be normal subgroup of G and $\alpha, \gamma \in \mathrm{Irr}(H)$. If $\alpha^G = e\chi$, for some integer e, and $[\gamma, (\chi\psi)_N] \neq 0$, then γ^G is a multiple of an irreducible. In particular, if $|G : N| = p$ then $\gamma^G \in \mathrm{Irr}(G)$.*

Theorem C. *Let p be a finite p-group and $\chi \in \mathrm{Irr}(G)$.*

(i) If $\chi \neq 1_G$ then $[\chi^2, \chi] = 0$

(ii) If $p \neq 2$ and $\chi(1) > 1$ then $[\chi^2, \lambda] = 0$ for all $\lambda \in \mathrm{Lin}(G)$.

(iii) Assume also that either $p \neq 2$ or $\eta(\chi^2) < p$. Then there exists a subgroup H of G and $\alpha \in \mathrm{Lin}(H)$ such that $\alpha^G = \chi$ and $(\alpha^2)^G \in \mathrm{Irr}(G)$. Thus χ^2 has an irreducible constituent of degree $\chi(1)$.

Let G be a p-group and $\chi \in \mathrm{Irr}(G)$. Assume that $\chi(1) = p^n$ with $n \geq 1$. Denote by $\overline{\chi}$ the complex conjugate of χ. In [2] it is proved that $\eta(\chi\overline{\chi}) \geq 2n(p-1)+1$.

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Thus $\eta(\chi\overline{\chi}) > p$. We can check that the principal character $1_G \in \mathrm{Irr}(G)$ is a constituent of $\chi\overline{\chi}$. Thus Theorem A (i) and (ii) may not hold true without the condition of $\eta(\chi\psi) < p$. Also, since $\chi(1) > 1$ and G is a p-group, there exist a subgroup N of G and $\alpha \in \mathrm{Irr}(N)$ such that $|G : N| = p$ and $\alpha^G = \chi$. Observe that $[1_N, (\chi\overline{\chi})_N] \neq 0$ and $\eta(1_N^G) = p$. Thus Theorem B may not hold true without the condition of $\eta(\chi\psi) < p$.

Consider the group $\mathrm{SL}(2,3)$ and the character $\chi \in \mathrm{Irr}(\mathrm{SL}(2,3))$ such that $\chi(1) = 3$. We can check that $[\chi^2, \chi] = 2$. Thus Theorem C (i) may not hold true without the condition of G being a p-group. Let G be the dihedral group of order 8 and $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 2$. We can check that

$$\chi^2 = \sum_{\lambda \in \mathrm{Lin}(G)} \lambda.$$

Thus Theorem C (ii) and (iii) may not hold true without the additional hypotheses.

2. Proofs

Let H be a subgroup of G and $\lambda \in \mathrm{Irr}(H)$. Denote by $\mathrm{Irr}(G \mid \lambda) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_H, \lambda] \neq 0\}$ the set of irreducible characters of G lying above λ.

Proof of Theorem A. **(i)** Observe that $\mathbf{Z}(\chi\psi) \leq \mathbf{Z}(\theta)$. Set $Z = \mathbf{Z}(\chi\psi)$. So $(\chi\psi)_Z = \chi(1)\psi(1)\gamma$ for some $\gamma \in \mathrm{Lin}(Z)$.

Suppose that $Z < \mathbf{Z}(\theta)$. Let Y/Z be chief factor of G such that $Y \leq \mathbf{Z}(\theta)$. Since Y/Z is a chief factor of a p-group, it is cyclic. Thus the character $\gamma \in \mathrm{Lin}(Z)$ extends to Y. We can check that

$$(\chi\psi)_Y = \frac{\chi(1)\psi(1)}{p} \sum_{\delta \in \mathrm{Lin}(Y \mid \gamma)} \delta.$$

If there exists some G-invariant δ in $\mathrm{Lin}(Y \mid \gamma)$, then all the elements in $\mathrm{Lin}(Y \mid \gamma)$ are G-invariant since G is a p-group and Y/Z is a chief factor. But then $\chi\psi$ must have at least p distinct irreducible constituents since $\theta_Y = \theta(1)\delta$ for some $\delta \in \mathrm{Lin}(Y \mid \gamma)$. Therefore the set $\mathrm{Lin}(Y \mid \gamma)$ forms a G-orbit. In particular $\mathbf{Z}(\theta)$ can not contain Y. We conclude that $Z = \mathbf{Z}(\theta)$.

(ii) Suppose that $\mathrm{V}(\chi\psi) \cap \mathrm{V}(\theta) < \mathrm{V}(\theta)$. Set $V = \mathrm{V}(\theta)$. We are going to conclude that necessarily we have that $\chi\psi$ has at least p distinct irreducible constituents.

Let V/W be a chief factor of G such that $W \geq \mathrm{V}(\chi\psi) \cap V$. Observe that if $g \in \mathrm{V}(\theta) \setminus W$, then $g \notin \mathrm{V}(\chi\psi)$. Therefore, by definition of $\mathrm{V}(\chi\psi)$, we have that $(\chi\psi)(g) = 0$ for all $g \in \mathrm{V}(\theta) \setminus W$. In particular, for all $\gamma \in \mathrm{Lin}(V/W)$ we have that

$$(\chi\psi)_V \gamma = (\chi\psi)_V. \tag{2.1}$$

Let $v \in V \setminus W$ such that $\theta(v) \neq 0$. Observe that such an element exist because otherwise $W = \mathrm{V}(\theta) = V$. Let $\sigma \in \mathrm{Irr}(V)$ be a constituent of $(\theta)_V$ and $\{\sigma^g \mid g \in T\}$ be the G-orbit of σ in G. By Clifford Theory we have that $\theta_V = e\sum_{g \in T} \sigma^g$, for some positive integer e. Thus

$$\sum_{g \in T} \sigma^g(v) = \frac{\theta(v)}{e} \neq 0. \tag{2.2}$$

Since G is a p-group and V/W is a chief factor of G, we have that G acts trivially on V/W. Therefore the set of characters $\mathrm{Lin}(V/W)$ are G-invariant. Thus the

G-orbit of $\sigma\gamma$ is the set $\{\sigma^g\gamma \mid g \in T\}$. By (2.1) $\sigma\gamma$ is an irreducible constituent of $(\chi\psi)_V$. Thus, there exists some character $\theta_\gamma \in \mathrm{Irr}(G)$ such that $[(\theta_\gamma)_V, \sigma\gamma] \neq 0$. By Clifford theory we have that $(\theta_\gamma)_V = f\sum_{g\in T}\sigma^g\gamma$ for some positive integer f. Therefore

$$\frac{\theta(v)}{e}\gamma(v) = [\sum_{g\in T}\sigma^g(v)]\gamma(v) = \sum_{g\in T}(\sigma^g\gamma)(v) = \frac{\theta_\gamma(v)}{f}. \tag{2.3}$$

We conclude that if $\gamma(v) \neq 1$, then $\theta(v) \neq \theta_\gamma(v)$ and therefore $\theta \neq \theta_\gamma$. Similarly we can check that if $\delta \in \mathrm{Lin}(V/W)$ and $\delta \neq \gamma$, then there exists a constituent $\theta_\delta \in \mathrm{Irr}(G)$ of the product $\chi\psi$ such that $[\delta, (\theta_\delta)_V] \neq 0$ and $\theta_\gamma \neq \theta_\delta$. Since $\mathrm{Lin}(V/W)$ has p distinct irreducible constituents, then $\chi\psi$ has at least p distinct irreducible constituents. A contradiction with our hypothesis. Therefore $\mathrm{V}(\theta) \leq \mathrm{V}(\chi\psi)$. $\square$

Lemma 2.4. *Let G be a finite p-group and N be a normal subgroup of G. Let $\phi \in \mathrm{Irr}(N)$. Then the set $\mathrm{Irr}(\,G \mid \phi\,)$ of all $\chi \in \mathrm{Irr}(G)$ lying over ϕ has either one or at least p members.*

Proof. Let G_ϕ be the stabilizer of ϕ in G. By Clifford theory we have that $\eta(\phi^{G_\gamma}) = \eta(\phi^G)$. If $|G_\phi| < |G|$, by induction we have that either $\eta(\phi^{G_\gamma}) = 1$ or $\eta(\phi^{G_\gamma}) \geq p$ and the result holds. We may assume that $|G_\phi| = |G|$. In Lemma 4.1 [1], it is proved that the result holds if ϕ is a G-invariant character. $\square$

Proof of Theorem B. Since N is normal in G and $\gamma^G = \chi$, the irreducible constituents of χ_N are of the form α^g, for some $g \in G$, and $(\alpha^g)^G = e\chi$. Since $[\gamma, (\chi\psi)_N] \neq 0$, there exist some $g \in G$ and some $\beta \in \mathrm{Irr}(N)$ such that $[\alpha^g\beta, \gamma] \neq 0$. By Exercise 5.3 of [3] we have that $(\alpha^g\psi_N) = (\alpha^g)^G\psi = e\chi\psi$. Thus the irreducible constituents of $(\alpha^g\psi_N)^G$ are irreducible constituents of $\chi\psi$. In particular, the irreducible constituents of $(\alpha^g\beta)^G$ are irreducible constituents of $\chi\psi$. Thus the irreducible constituents of γ^G are irreducible constituents of $\chi\psi$. By Lemma 2.4, either $\eta(\gamma^G) = 1$ or $\eta(\gamma^G) \geq p$. Since $\eta(\chi\psi) < p$, it follows that $\eta(\gamma^G) = 1$, i.e γ^G is a multiple of an irreducible. $\square$

Proof of Theorem C. **(i)** Observe that $[\chi^2, \chi] = [\chi, \chi\overline{\chi}]$. Observe also that $\mathrm{Ker}(\chi\overline{\chi}) = \mathbf{Z}(\chi)$. Thus if $[\chi, \chi\overline{\chi}] \neq 0$, then $\mathrm{Ker}(\chi) \geq \mathbf{Z}(\chi)$. Therefore $\mathrm{Ker}(\chi) = \mathbf{Z}(\chi)$. Since $\mathbf{Z}(G/\mathrm{Ker}(\chi)) = \mathbf{Z}(\chi)/\mathrm{Ker}(\chi)$ and G is a p-group, it follows that $G = \mathrm{Ker}(\chi)$ and so $\chi = 1_G$.

(ii) Assume that $[\chi^2, \lambda] \neq 0$ for some $\lambda \in \mathrm{Lin}(G)$. Since $p \neq 2$ and G is a p-group, there exists some character $\beta \in \mathrm{Lin}(G)$ such that $\beta^2 = \lambda$. Thus $[\chi^2, \lambda] = [\chi^2, \beta^2] = [\chi\overline{\beta}, \overline{\chi}\beta]$. Since $\chi\overline{\beta}, \overline{\chi}\beta \in \mathrm{Irr}(G)$, it follows that

$$\chi\overline{\beta} = \overline{\chi}\beta = \overline{\chi\overline{\beta}}.$$

Since $\chi\overline{\beta} \in \mathrm{Irr}(G)$ is a real character and G is a p-group with $p \neq 2$, it follows that $\chi\overline{\beta} = 1_G$. Thus $\chi = \beta$ and $\chi(1) = 1$.

(iii) Observe that $\mathrm{Ker}(\chi^2) \geq \mathrm{Ker}(\chi)$. Working with the group $G/\mathrm{Ker}(\chi)$, without lost of generality we may assume that $\mathrm{Ker}(\chi) = 1$. We may also assume that $\chi(1) > 1$. We are going to use induction on the order of $|G|$.

Let $Z = \mathbf{Z}(\chi)$ be the center of the character χ. Let Y/Z be a chief factor of G. Let $\zeta \in \mathrm{Lin}(Z)$ be the unique character of Z such that $\chi_Z = \chi(1)\zeta$. Since G is a p-group and Y/Z is a chief factor, it follows that ζ extends to Y. Let $\iota \in \mathrm{Lin}(Y)$ be an extension of ζ. Since $Z = \mathbf{Z}(\chi)$ and Y/Z is a chief factor of a p-group, it follows that ι lies below χ. Let G_ι be the stabilizer of ι in G. Observe that the

stabilizer G_{ι^2} of ι^2 contains G_ι. Since $|G : G_\iota| = p$, either $G = G_{\iota^2}$ or $G_{\iota^2} = G_\iota$. If $\eta(\chi^2) < p$, by Theorem A (i) we have that $G_{\iota^2} = G_\iota$. If G is a p-group with $p \neq 2$, then $(\iota^g(y))^2 = (\iota(y))^2$ implies that $\iota^g(y) = \iota(y)$. Thus $G_{\iota^2} = G_\iota$ if $\eta(\chi^2) < p$ or $p \neq 2$.

Let $\chi_\iota \in \mathrm{Irr}(G_\iota)$ be the Clifford correspondent of χ and ι, i.e. $\chi_\iota^G = \chi$ and χ_ι lies above ι. Observe that $(\chi_\iota^2)_Y = \chi_\iota^2(1)\iota^2$ and $\iota^2 \in \mathrm{Lin}(Y)$. Thus all the irreducible constituents of the character χ_ι^2 lie above ι^2. Since $G_\iota = G_{\iota^2}$, by Clifford theory all the irreducible constituents of χ_ι^2 induce irreducibly to G. Since $|G_\iota| < |G|$, by induction there exist a subgroup H of G_ι and a character $\alpha \in \mathrm{Lin}(H)$ such that $\alpha^{G_\iota} = \chi_\iota$ and $(\alpha^2)^{G_\iota} \in \mathrm{Irr}(G_\iota)$. Since all the irreducible constituents of χ_ι^G induce irreducibly, it follows that $(\alpha^2)^G \in \mathrm{Irr}(G)$. Since $\chi_\iota^G = \chi$, it follows that $(\alpha)^G = \chi$. Since $[(\chi^2)_H, \alpha^2] \neq 0$ and $(\alpha^2)^G \in \mathrm{Irr}(G)$, it follows that $(\alpha^2)^G$ is an irreducible constituent of χ^2. Therefore χ^2 has an irreducible constituent of degree $\chi(1)$. □

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